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EXPERIMENTAL INVESTIGATION OF THE FUNDAMENTAL MODES OF A COLLISIONLESS PLASMA

by

David L. Book, John H. Malmberg, and Charles B. Wharton

ANNUAL SUMMARY REPORT
March 10, 1964 through March 31, 1965

Contract No. NAS7-275
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Western Operations Office
Santa Monica, California

May 17, 1965

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During the period of this report, the following "reportable items" as defined by the article "Report of New Technology" evolved:"

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Space Administration
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FOREWORD

This annual summary report describes work performed from March 10, 1964, through March 31, 1965, on NASA Contract NAS7-275, "Experimental Investigation of the Fundamental Modes of a Collisionless Plasma." The General Atomic principal investigator is Dr. John H. Malmberg and the General Atomic project number is 407. The project monitor is Dr. Karlheinz Thom.

ABSTRACT

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The dispersion relations for the electrostatic waves in collisionless plasma have been measured near the electron cyclotron frequency. Several waves are observed, including the C_{01} backward wave between the cyclotron frequency and the upper hybrid frequency and the plasma perturbed $T_{e_{11}}$ wave guide mode. The cyclotron waves are heavily damped. The experimental techniques necessary for the measurement are described. The Landau damping of the electrostatic modes of a plasma with finite radial dimensions is investigated theoretically. The growth of these electrostatic modes due to a beam whose energy depends on radius is also calculated.

Author

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I

INTRODUCTION

This report covers both the development of experimental techniques necessary for the investigation of the fundamental modes of a collisionless plasma and the theoretical and experimental results obtained. It is appropriate to evaluate the accomplishments of the project, by comparing the results obtained with the original statement of work for the contract, namely:

1. To measure the wavelength as a function of frequency of plasma waves near the electron cyclotron frequency;
2. To measure spatial attenuation of these waves;
3. To determine the effect on these waves as the plasma properties are varied systematically; and
4. To develop theoretical dispersion relations for plasma waves in non-uniform plasmas.

The experimental techniques necessary to measure the dispersion relations of the plasma waves have been developed, and the measurements called for in items 1 and 3 have been made. A preliminary report of this work has been given at a scientific meeting (see Appendix I), and these experimental results will be the subject of a scientific publication in the immediate future. The spatial attenuation of these waves has been measured, but we do not yet consider these measurements satisfactory, principally because of interference from a number of other waves in the same frequency range. We have measured dispersion curves for these other waves, in addition to those for the cyclotron wave we originally intended to investigate. We have also begun the study of low-level low-frequency turbulence in the machine, which interferes somewhat with the wave transmission measurements. This turbulence is also worth investigating because of its production of anomalous diffusion, an important phenomena in many applications. Experimental results are described in some detail in Section III of this report.

The theoretical work has concentrated on developing dispersion relations (including the "imaginary part" which predicts wave growth and damping) for plasma waves in non-uniform plasmas. Most of the work has been based on the plasma geometry obtained in our machine, i.e., a long column, of finite radius, with a smoothly varying, radial density profile. The results are described in Section II of this report, while the detailed calculations are presented in Appendices II and III. A paper reporting this work is in preparation and will be submitted for publication in the scientific literature.

The scientific objectives for the first year of the project have been substantially obtained. The work has also led to a number of interesting new problems which, together with refinement of the results already obtained, will be pursued during the current one-year extension of this program.

II

THEORETICAL RESULTS

The principal theoretical problems arising in connection with the cyclotron wave measurements are associated with deriving expressions, in a range of parameters appropriate for the machine, for the dependence of frequency on wave number and for the growth and damping rates of the waves. In particular, it is desirable to predict the effects on the dispersion relation of finite plasma temperature and of a radial variation in density for both the upper and lower branch space-charge waves.

We have obtained a general expression for the decay rate in time of both modes as a result of Landau damping in the presence of a radial density gradient*:

$$\frac{\gamma}{\omega_o} = \frac{\frac{1}{4}\frac{\sqrt{\pi}\,\omega_o}{k_\parallel v_T}\exp\left[-\left(\frac{\omega_o-\Omega_e}{k_\parallel v_T}\right)^2\right]F + \frac{\sqrt{\pi}\,\omega_o^3}{k_\parallel^3 v_T^3}\exp\left[-\left(\frac{\omega_o}{k_\parallel v_T}\right)^2\right]G}{\frac{\omega_o^4}{(\omega_o^2-\Omega_e^2)^2}F + G} \; ; \qquad (1)$$

where

$$F = \int_0^\infty dr\; r\, \omega_p^2 \left|\frac{d\Psi}{dr}\right|^2$$

and

$$G = k_\parallel^2 \int_0^\infty dr\; r\, \omega_p^2 \left|\Psi\right|^2$$

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*See also Appendix II of this report.

depend on measured density and potential profiles. This result reduces to the usual expression for Landau damping at long wave lengths and low frequency (lower branch), and to that for cyclotron damping for higher frequencies (upper branch). It is to be noted that the presence of a density gradient is essential to the latter result; in a homogeneous plasma, a longitudinal wave is not damped at the cyclotron frequency.

If a monoenergetic beam of electrons is injected into the experimental apparatus from the end, the electrons will acquire a radially dependent energy spectrum in the plasma due to the local radially dependent electric potential. This effect can be utilized to produce a finite growth rate for lower branch waves having the proper phase velocity. By choosing beam density and energy properly and "shaping" the beam with a template at the injection point, it is possible to introduce a growth rate in a variety of forms, in particular the "gentle bump" instability of quasi-linear theory. Without this energy spreading, the injected velocity distribution is a delta function and produces a "two-stream" instability. The non-linear effects associated with the delta function distribution are much harder to analyze than those associated with the "gentle-bump" case. The analysis proceeds from the result of Eq. (1). (See also Appendix III.)

Preliminary calculations indicate that the results of Trivelpiece and Gould[1] for finite-size effects on the upper branch of the ω, k plot must be generalized in the case of a smoothly varying radial density. An investigation of the finite temperature dependence of the upper branch and an analysis of the mechanisms of propagation and decay for this branch in greater detail is also planned.

III
EXPERIMENTAL RESULTS

At the beginning of this program, there already existed at General Atomic a machine for producing a plasma suitable for these experiments. In addition, the so-called "lower branch" space charge waves had been extensively investigated under other support. Descriptions of the machine[2] and of the measurements on the lower branch wave[3,4] have been published. However, to measure the dispersion of the cyclotron waves, the diagnostic techniques needed to be improved. Much greater sensitivity is necessary to observe this wave than is necessary for the lower branch, for reasons which are not yet clearly understood. This difficulty has been encountered by experimentalists elsewhere and in the past has prevented the tracing out of cyclotron wave-dispersion curves. Therefore, the first few months of the experimental effort were largely devoted to engineering and diagnostic improvements.

The experimental procedure is as follows: Two probes, each consisting of a single radial wire about 8 cm long, are placed in the plasma in the central portion of the experimental apparatus. One probe is connected to a transmitter and the other, by coaxial cables, to a receiver. By comparing the phase of the received signal with that of the transmitted signal (with the use of a microwave interferometer), the phase shift in the plasma may be determined as a function of probe separation, and thus the wavelength may be measured. A measurement of the amplitude of the received wave versus position gives the damping length. The first step for the present program was to redesign and rebuild the probe manipulators, which were not sufficiently sophisticated for this work, and to transduce their position for direct plotting. The probes may now be radially retracted 3-1/2 in., and a transducer has been installed on this motion so that floating potential, saturation currents, and wave power may be automatically plotted on the x-y recorder as a function of radius. The probe may still be moved longitudinally the full length of the machine, and this motion is also transduced. The new manipulators have been improved mechanically to eliminate the binding and jamming troubles we have previously had with the longitudinal motion. The accuracy of the angular manipulator has also been greatly improved.

A difficulty with a transmission experiment of this kind has been the problem of sorting out plasma wave effects from direct electromagnetic coupling between the probes. We have achieved a very large reduction in electromagnetic coupling by surrounding the plasma with a stainless steel tube designed as a waveguide beyond cutoff. We have tried tubes of various sizes from 3 in. to 6 in. For the 3 in. tube, we obtain an attenuation of about 9 dB/cm of separation, which is satisfactory for our purposes. As the tube is slotted almost full length, the probes can still be moved longitudinally.

For our first transmission experiments, we used a swept-frequency transmitter and a wide-band receiver which displayed the detected signal on an oscilloscope. A system of this type does not have sufficient sensitivity or signal-to-noise ratio to permit observation of the transmission of heavily damped waves. Since much of our interest is in the transmission and damping of waves that are heavily damped, we have had to devise a more sensitive detection scheme. We now use a transmitter set at a single rf frequency and chopped at a few kilocycles. The receiver includes a sharp, high-frequency filter, a string of broad-band amplifiers, an rf detector, a video amplifier, and a coherent detector operated at the transmitter chopping frequency. The rf bandwidth is thus a few hundred kilohertz, and the video bandwidth of the coherent detector is a few hertz. The resultant improvement in the signal-to-noise ratio increases the usable sensitivity of the receiver by 40 or 50 dB.

An even greater improvement in signal-to-noise ratio may be obtained by measuring the amplitudes of the peaks of the interferometer output. In this case, the phase reference path for the interferometer is not chopped, but the rf signal to the transmitting antenna is. The phase reference signal is made large enough to operate the crystal detector well above the insensitive, square-law region. That part of the output signal from the interferometer crystal which is coherent with the transmitter chopping frequency is time-averaged and detected. This system is, in effect, a double coherent-detector. The interferometer sorts out signals at the rf frequency of the transmitter and in phase with it, with a bandwidth about equal to the chopper frequency, i.e., $\sim$ 1 kHz. That part of the envelope of such signals that agrees in frequency and phase with the transmitter chopping frequency is time-averaged and displayed on the x-y recorder. Thus, the system detects coherently at the rf frequency and again at the chopper frequency, resulting in an ultimate bandwidth, determined by the final averaging time, of a few hertz. This bandwidth is sufficiently small that plasma noise is negligible in the final received signal. All of the signal seen comes from the transmitter and is due to transmitted waves. It is clear, however, that while this scheme eliminates confusion with noise in the receiver, it does not eliminate the influence of the noise spectrum in the plasma on the dynamics of wave transmission. That is to say, those processes in the plasma which generate noise on the receiving probe in the absence of a transmitted signal also modify the wave transmission. We will return to this point presently, and this modulation will be observed in the received signal.

We have made a series of measurements with Langmuir probes on the geometry and parameters of the plasma. In a typical case, we have a radius of 7mm, a length of 230 cm, a density of 5×10^8 electrons cm^{-3}, a temperature of 12 ± 3 eV, and a neutral density of 5×10^{11} molecules cm^{-3} (mostly H_2). The density may be varied from about 1×10^8 to about 3×10^9 by varying the duoplasmatron arc current. Hence we are able to set the plasma frequency either above or below the cyclotron frequency. The plasma density falls about a factor of 10 in the length of the machine, because ions are migrating radially across the magnetic field and striking the shield.

For early transmission experiments near the electron cyclotron frequency, a slow-wave helix surrounding the plasma column was used to launch the waves. This geometry should preferentially couple to plasma waves of short wave length. The helix was mounted on the ion trap and surrounded by a wave-guide-beyond-cutoff structure for electromagnetic shielding. The receiving probe was a small-diameter transverse wire.

The dispersion relation we observed for waves near electron cyclotron frequency is shown in Figure 1. The plasma frequency in this case is less than the electron cyclotron frequency but not small compared with it. The low-frequency branch of the curve is also given in Figure 1. The upper branch of the dispersion curve may be represented by the formula

$$f - f_c = \pm v_p/\lambda \quad , \tag{2}$$

where

f = the transmitter frequency,
f_c = the electron cyclotron frequency,
λ = the wave length of the transmitted wave, and
$v_p = 3.5 \times 10^8$ cm-sec^{-1}, for the case exhibited in Figure 1.

The dispersion relation of this wave is that expected if the plasma distribution function has an electron beam in the tail, with a velocity equal to v_p. These waves appear to be resonant cyclotron waves on an electron beam in the plasma. We have been able to show that these beams were due to secondary electrons generated by ions plowing into the wave-launching helix. In particular, the group velocity of these waves can be accurately computed from the difference in potentials of the helix and the plasma. That is, the group velocity is just equal to the velocity of the secondary electrons in the plasma. When the ion trap potential is changed, the group velocity changes correspondingly.

The existence of these beam-waves in the plasma is an interesting effect, but not the one we set out to look for, so we have eliminated them by removing the helix and using probes to launch the waves. This reduced

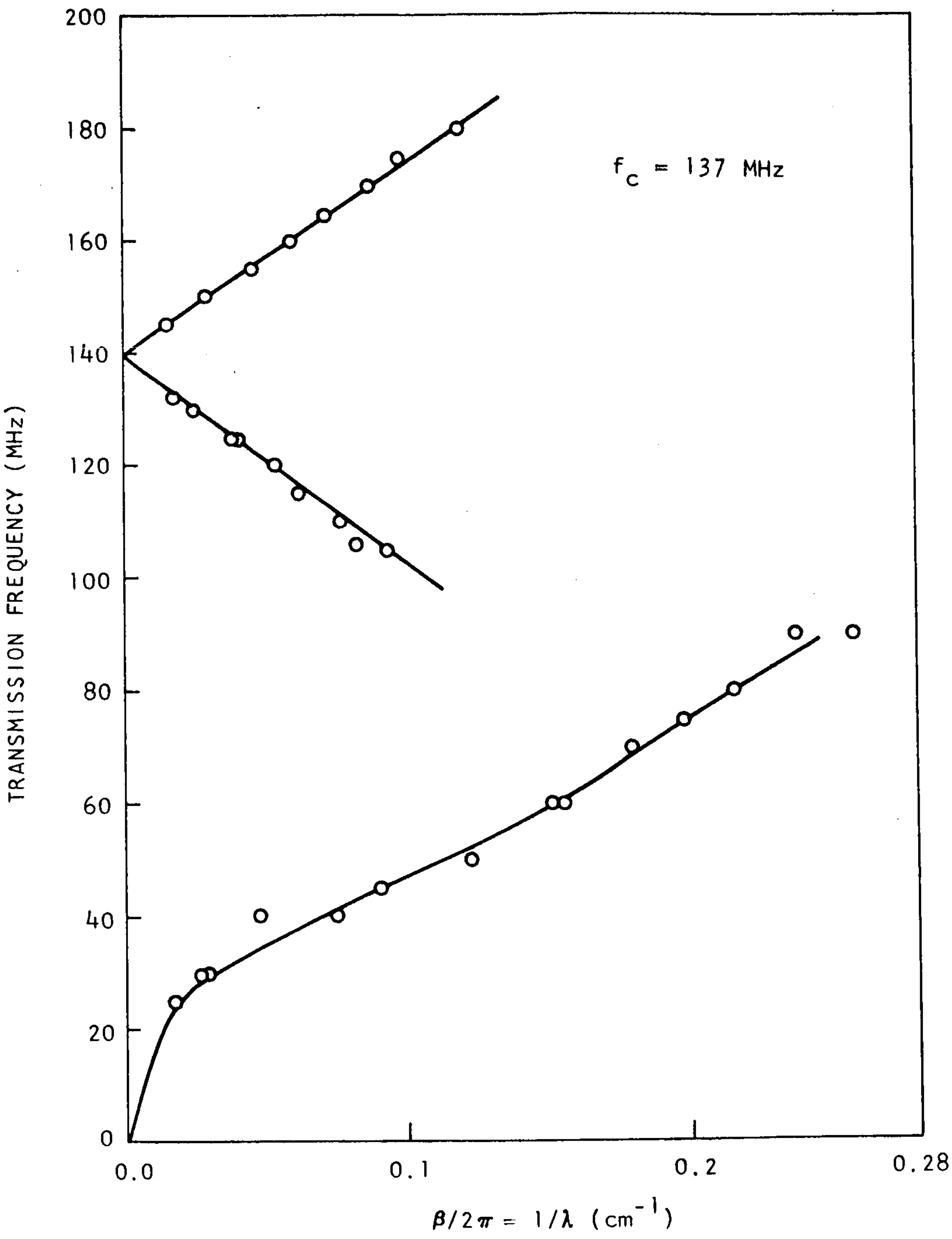

Figure 1--Dispersion relation

the wave launching efficiency, but by using improved instrumentation we have still retained sufficient resolution to find cyclotron waves. Secondary electrons from the ion-trap suppressor grid are still a minor problem for wave propagation in the upstream direction, leading to abnormal damping and, in some cases, even wave growth. Their current density is too low to cause an instability, and no effect on downstream propagation is evident. At least three (and perhaps more) distinct waves having resonances (wave-number becoming large) at or near the electron cyclotron frequency have been catalogued. Two waves lie above the cyclotron frequency. One has a velocity $v_\varphi > c$ and is a forward wave. The other is a "slow wave", having a phase that retards with frequency, i.e., a backward wave. The fast wave is apparently an electromagnetic waveguide mode, perturbed by the plasma. The slow wave is apparently the C_{01} cyclotron wave,[1,5,6] having a propagation cutoff at the upper hybrid frequency, $f_{uh} = (f_b^2 + f_p^2)^{\frac{1}{2}}$. Typically, the damping is 10 to 30 dB per wavelength. Dispersion curves are shown in Figures 2, 3, and 4. Only the "downstream" half of the ω - β diagrams is plotted, to avoid any possible streaming effects of secondary electrons.

Below the cyclotron frequency, there appear to be several waves. The fastest of these, fairly certainly, is a plasma-perturbed TE_{mn} waveguide mode, mentioned above. Its dispersion curve matches that for a TE_{11} mode in a waveguide whose cross-section is 1/10-filled with plasma. We have a plasma column of about 2-cm diameter inside a 12-cm diameter tube, which is a reasonable fit, if the radial density gradients are accounted for. The dispersion for this wave is very similar to that for a whistler.[7,8]

The other "cyclotron waves" may be higher modes, that is, those having circumferential field variations, but because of standing wave problems and heavy damping, there is considerable uncertainty in interpretation of our presently accumulated data. Recent improvements of the system may permit clarification of these uncertainties. If any of these waves do represent higher mode cyclotron waves, they should exhibit Faraday rotation, and we plan to look for this effect. Unfortunately, the waves are heavily damped and can be followed for only a wavelength or two, making measurements very difficult. The "whistler" mentioned above would also have Faraday rotation, except that the ordinary wave component is cut off by the waveguide cutoff, leaving only the elliptically polarized extraordinary component. No evidence for polarization rotation of this wave has been found in our experiment.

The "lower branch" waves--the conventional spacecharge waves of a warm, finite plasma column--have been extensively investigated at General Atomic and elsewhere.[1,3,5,7] We have plotted the measured dispersion curves here, since we use these waves in a diagnostic manner to aid us in understanding the cyclotron wave characteristics.

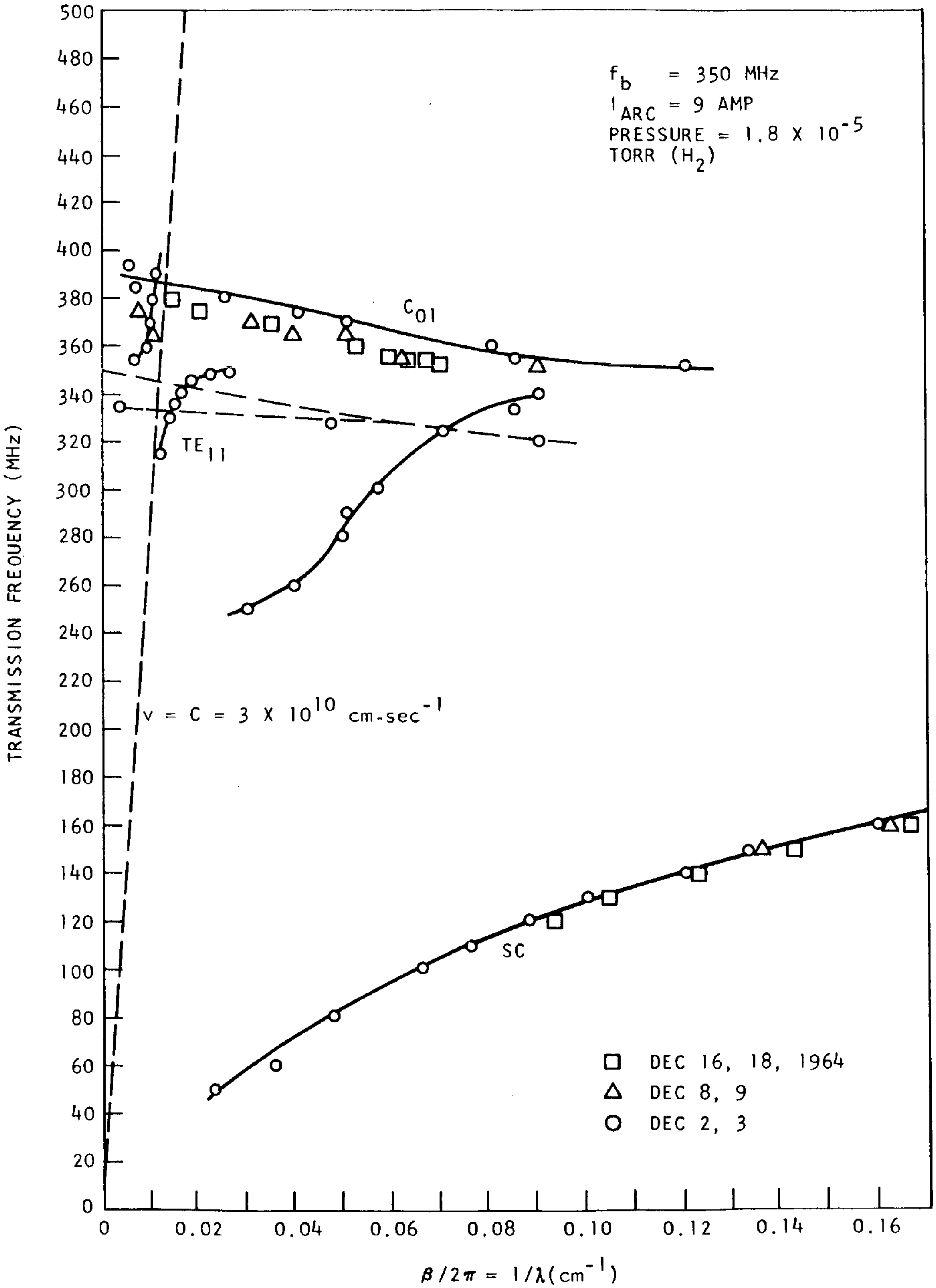

Figure 2--ω-β diagrams for waves in the collisionless H_2 plasma, over the frequency range 0-500 MHz. Cyclotron frequency f_b was 350 MHz, and plasma density n_e was approximately 4.2×10^8 cm^{-3}

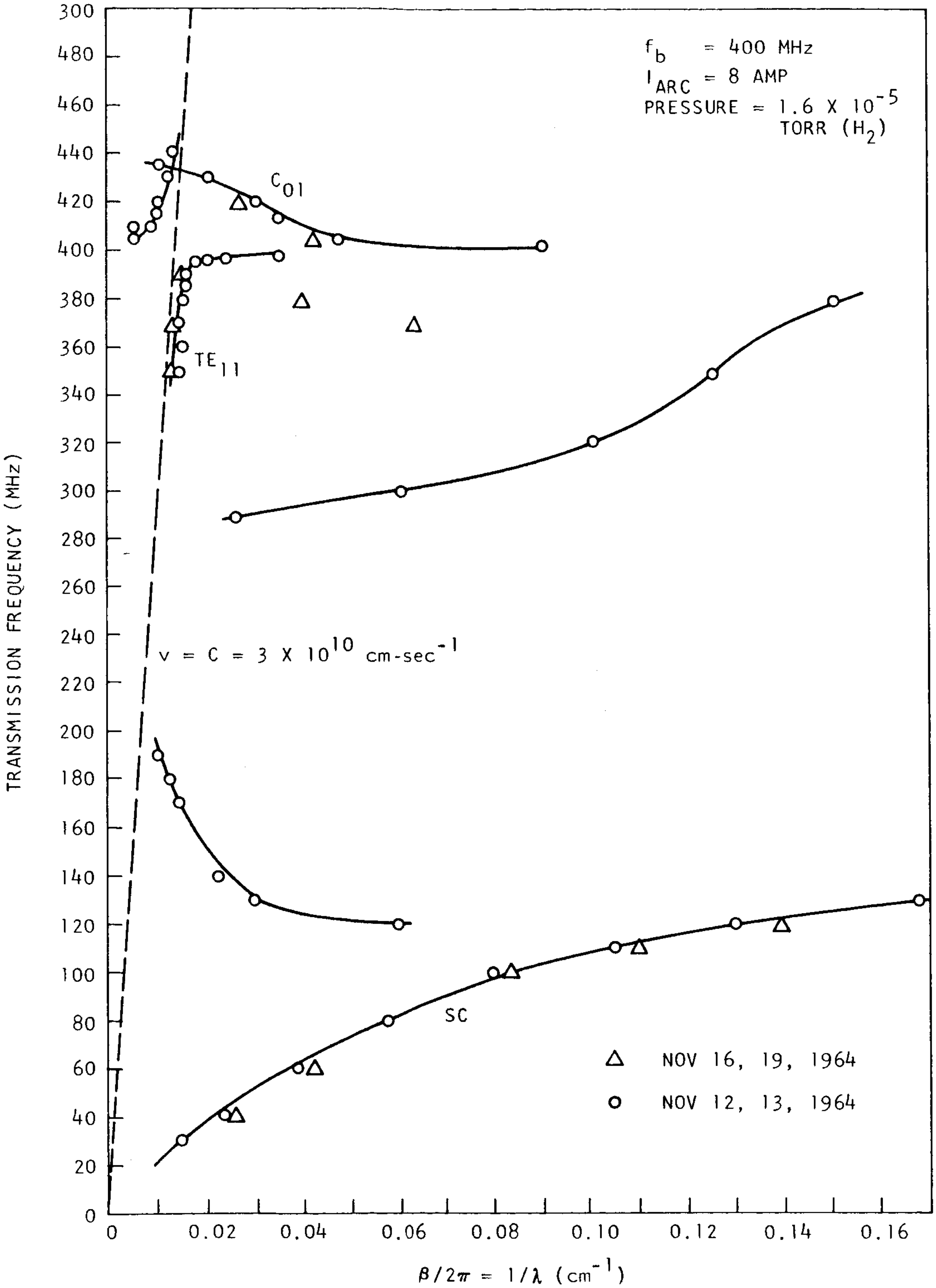

Figure 3--ω-β diagrams as in Figure 2, but f_b = 400 MHz, $n_e \cong 3.9 \times 10^8$ cm^{-3}

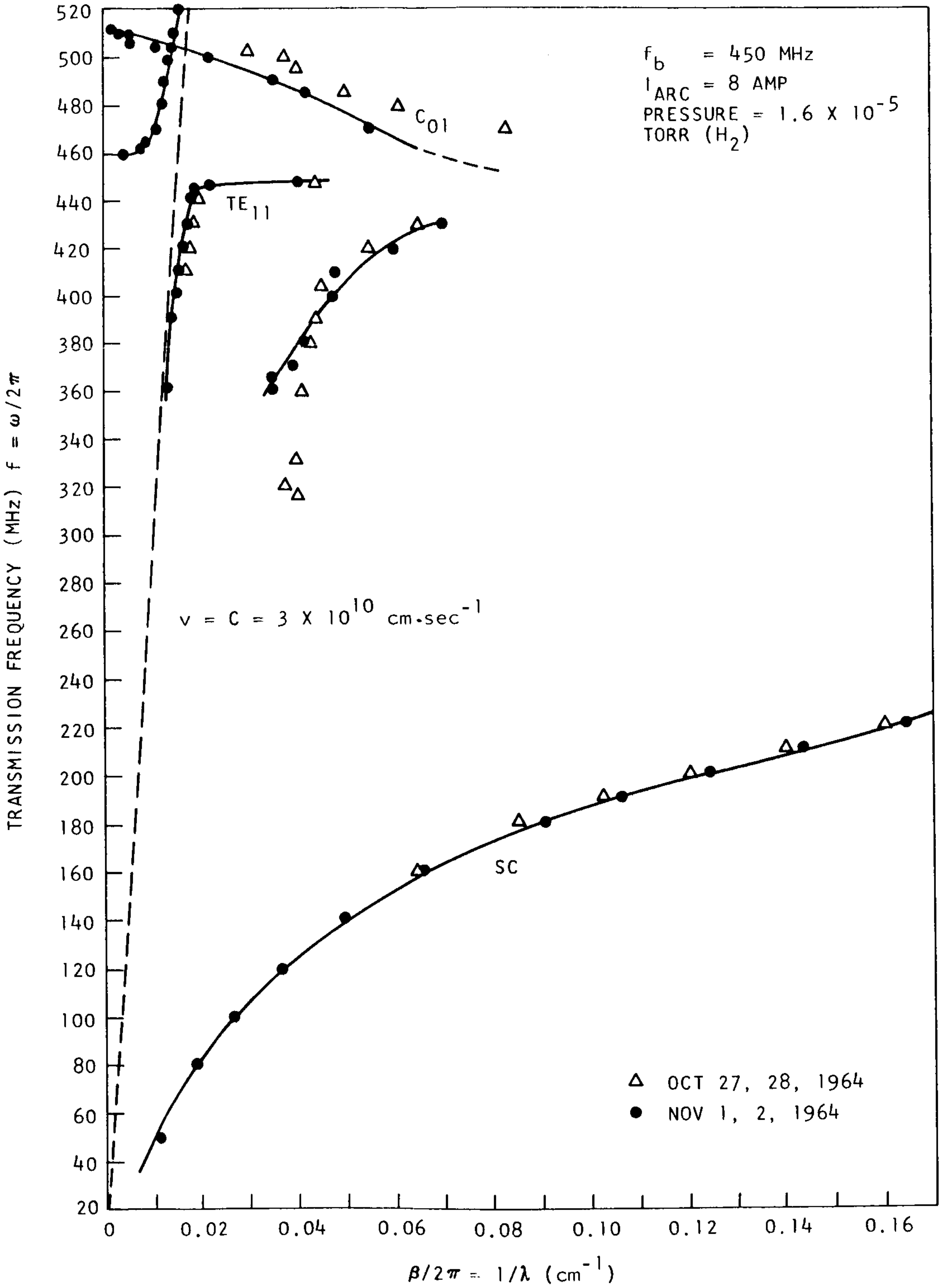

Figure 4--ω-β diagrams as in Figure 2, but f_b = 450 MHz, $n_e \cong 7.1 \times 10^8$ cm^{-3}

Figures 2, 3, and 4 show effects of various plasma densities and cyclotron frequencies on the cyclotron family of waves. We have also made a preliminary study of some of the effects of various electron temperatures, by adding an impurity of argon to the hydrogen in an attempt to decrease T_e from its usual 6 to 12 eV down to ≈ 5 eV. Evidence for the occurrence of a temperature decrease is meager, but seems encouraging. The (collisionless) damping of the lower branch, which is a measure of the plasma temperature, can be decreased drastically, at any particular frequency and wavenumber, by empirically adjusting the ratio of the H_2 and A partial pressures. The data plotted in Figure 5 were obtained for H_2 in the plasma source in the normal manner. A frequency of 240 MHz, yielding a wavelength of 2 cm, was chosen. The H_2 partial pressure was decreased from 1.8 to 1.4×10^{-5} torr. The waves vanished. Argon was then added to the plasma source until the waves returned, and the pressure was adjusted to yield the same plasma wavelength of 2 cm. The wave attenuation was found to have decreased from 5.1 dB per wavelength to 3.6 dB per wavelength. This corresponds to a ratio of electron temperatures of ≈ 0.4. A direct measurement of the slope of the electron-energy distribution function, using an electron-energy analyzer mounted behind the ion trap, gave a ratio of ≈ 0.7. The partial-dispersion curve taken with argon (shown in Fig. 6) has an unusual curvature and thus is somewhat suspect. We plan to continue the argon admixture experiments on the lower branch, with slightly improved instrumentation and in a more quantitative fashion. We will then look at the effects on the upper branch (cyclotron family) resulting from the addition of argon or other gas, to see if there is also collisionless damping of these waves, and if so, whether it can be decreased at large wavenumbers by decreasing T_e.

During the wave transmission experiments, we have observed that the received signal exhibits strong noise modulation in the frequency range of a few tens of kilohertz. As previously pointed out, this noise is not directly observed by our exceedingly narrow band receiver, but the transmission is nevertheless amplitude-modulated by it. While the plasma we are using for these experiments is very quiet by usual experimental standards, it is certainly very much noisier than a theoretical plasma in thermal equilibrium. A quieter plasma would make the experiments easier. The plasma also has a density dependence in the longitudinal direction, due to the fact that the ions are diffusing radially as they drift through the machine. A careful analysis indicates that this diffusion is much too rapid to be caused by binary collisions and must be due to some anomalous diffusion process. This kind of enhanced diffusion is of great interest for practical problems. For these two reasons, we have undertaken to study in some detail the nature of the noise in the machine in this frequency range. We have been able to establish that the plasma exhibits a fully developed instability, or what should perhaps be called a stable dynamic motion. The plasma rotates with a small amplitude around the symmetry axis of the

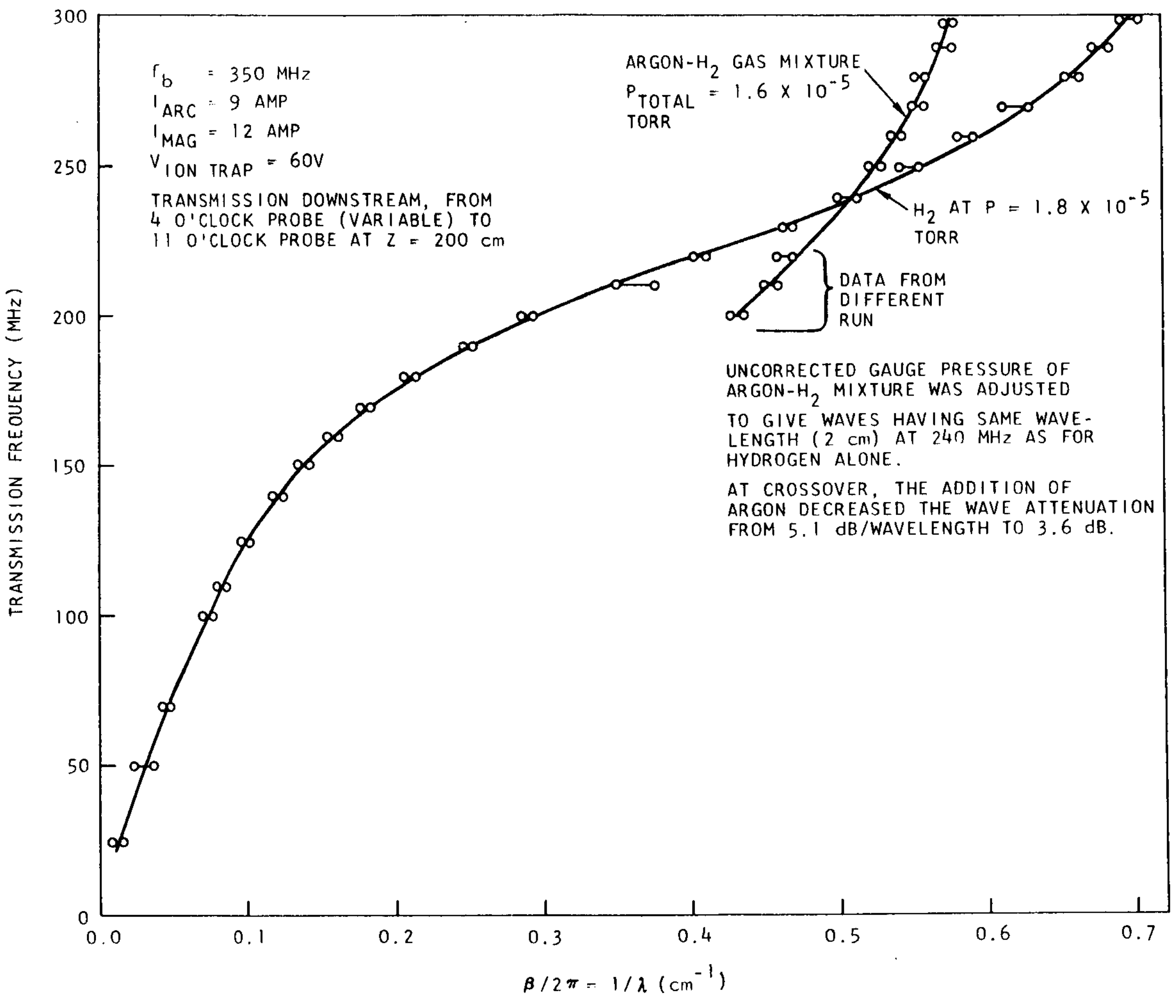

Figure 5--ω-β diagram for spacecharge waves in a collisionless plasma in hydrogen and in a mixture of hydrogen and argon

machine, with a frequency in the range 30 to 50 kH_z. The rotation is in the E × B direction, and the frequency is consistent with the radial electric field produced by the plasma potential and the longitudinal magnetic field due to the main coils. In the language of instability analysis, this is probably a fully developed m = 1 flute instability. We have tried a number of schemes for shutting off this instability. One that appears to work is a rearrangement of the machine geometry to make the plasma collision-dominated for a few inches at the source end. Presumably, this short circuits electric fields across the magnetic field lines. This short circuiting is theoretically known to stabilize flutes. These experiments are still in progress and not yet definitive. It does not appear to be absolutely necessary to suppress this instability in order to extend our measurements on the cyclotron waves. However, we would expect that reducing the effect would make these experiments easier, and we intend to pursue this approach somewhat further.

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APPENDIX I

CYCLOTRON WAVES IN A COLLISIONLESS PLASMA[†]

Abstract

The propagation characteristics of spacecharge waves in a long collisionless plasma column[1] have been studied over the frequency range 90 to 520 MHz. The collisionless damping of these waves has previously been reported.[2] Recently the growth of the wave amplitudes in space, due to an injected electron beam having a velocity spread,[3] has been investigated. In this paper, we report the appearance of a wave at frequencies slightly above the cyclotron frequency ω_b. The wave appears to be longitudinal, with velocities ranging from 3×10^8 to 10^{10} cm-sec^{-1} over the frequency range. There is also a wave at frequencies slightly below ω_b, which seems to be directed by the plasma column and which may be a whistler.

[†] Wharton, C. B., and J. H. Malmberg, Paper presented at APS meeting, Division of Plasma Physics, New York, November 4-7, 1964. Abstract published in APS Bull., 10, 199 (1965).

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APPENDIX II

LANDAU DAMPING OF ELECTROSTATIC MODES WITH FINITE SYSTEM EFFECTS

The following situation is to be considered: A plasma is confined by a finite magnetic field directed along the z-axis. We are interested in two cases, those of (a) two-dimensional (slab) and (b) cylindrical geometry. Assume that the cross-sectional electron distribution n drops off as some smooth function (e.g., Gaussian) away from the center of the plasma. It is required to find the damping rates γ for longitudinal modes propagating in this system as a functional of the measured density profile and electric potential.

We start with Laplace's equation

$$\nabla \cdot \underset{\approx}{\epsilon} \cdot \nabla\varphi = 0 \quad . \tag{1}$$

In case (a), for the potential, we write

$$\varphi = \Psi(x)\, e^{ik_{\parallel} z - i\omega t} \quad . \tag{2}$$

We assume Maxwellian velocity distributions with ion temperatures not substantially larger than electron temperatures, so only the electron contribution to $\underset{\approx}{\epsilon}$ need be retained. Then in the long wavelength approximation,

$$\frac{\omega}{k_{\parallel}} >> v_T \quad , \quad \frac{\omega}{k_{\perp}} >> v_T \quad ; \tag{3}$$

$\underset{\approx}{\epsilon}$ is given by[1]

$$\underset{\approx}{\epsilon} = \hat{x}\hat{x}\,\epsilon_{\perp} + \hat{z}\hat{z}\,\epsilon_{\parallel} \quad , \tag{4}$$

where

$$\epsilon_{\parallel} = 1 + \sum_{n=-\infty}^{\infty} \frac{2\omega_p^2}{\omega k_{\parallel} \nu_T} e^{-z} I_n(z)\, x_n \left[1 + x_n Z(x_n)\right]$$

$$\approx 1 + \frac{2\omega_p^2}{\omega k_{\parallel} \nu_T} \frac{\omega}{k_{\parallel} \nu_T} \left\{ 1 + \frac{\omega}{k_{\parallel} \nu_T} \left\{ - \frac{k_{\parallel} \nu_T}{\omega} - \frac{1}{2} \left(\frac{k_{\parallel} \nu_T}{\omega}\right)^3 + i\sqrt{\pi}\, \exp\left[-\left(\frac{\omega}{k_{\parallel} \nu_T}\right)^2\right] \right\}\right\}$$

$$= 1 - \frac{\omega_p^2}{\omega^2} + i2\sqrt{\pi}\, \frac{\omega_p^2 \omega}{\left(k_{\parallel} \nu_T\right)^3} \exp\left[-\left(\frac{\omega}{k_{\parallel} \nu_T}\right)^2\right] ; \qquad (5)$$

$$\epsilon_{\perp} = 1 + \sum_{n=-\infty}^{\infty} \frac{\omega_p^2}{\omega k_{\parallel} \nu_T} Z(x_n)\, e^{-z} \frac{n^2}{z} I_n(Z)$$

$$\approx 1 + \frac{1}{2} \frac{p^2}{\omega k_{\parallel} \nu_T} \left[Z\left(\frac{\omega - \Omega_e}{k_{\parallel} \nu_T}\right) + Z\left(\frac{\omega + \Omega_e}{k_{\parallel} \nu_T}\right)\right]$$

$$\approx 1 + \frac{1}{2} \frac{\omega_p^2}{\omega k_{\parallel} \nu_T} \left\{ - \frac{k_{\parallel} \nu_T}{\omega - \Omega_e} + i\sqrt{\pi}\, \exp\left[-\left(\frac{\omega - \Omega_e}{k_{\parallel} \nu_T}\right)^2\right] - \frac{k_{\parallel} \nu_T}{\omega + \Omega_e} + i\sqrt{\pi}\, \exp\left[-\left(\frac{\omega + \Omega_e}{k_{\parallel} \nu_T}\right)^2\right]\right\}$$

$$= 1 - \frac{\omega_p^2}{\omega^2 - \Omega_e^2} + i\, \frac{1}{2} \frac{\sqrt{\pi}\, \omega_p^2}{\omega k_{\parallel} \nu_T} \left\{ \exp\left[-\left(\frac{\omega - \Omega_e}{k_{\parallel} \nu_T}\right)^2\right] + \exp\left[-\left(\frac{\omega + \Omega_e}{k_{\parallel} \nu_T}\right)^2\right]\right\} . \qquad (6)$$

Here ω_p is the electron plasma frequency, $\omega_p^2 = (4\pi e^2/m)\, n(x)$; Ω_e is the electron cyclotron frequency; $\nu_T^2 = 2KT/m$;

$$z = \frac{K_\perp^2 KT}{\Omega_e^2 m} << 1 \quad ;$$

$$x_n = \frac{\omega - n\Omega_e}{k_\parallel \nu_T} >> 1 \text{ (unless } \omega \approx n\Omega_e\text{), by Eq. (3)} \quad ;$$

Z is the plasma dispersion function of Fried and Conte,[2] defined by

$$Z(\zeta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx \, \frac{e^{-x^2}}{x - \zeta}$$

$$\rightarrow i\sqrt{\pi}\, e^{-\zeta^2} - \frac{1}{\zeta}\left(1 + \frac{1}{2\zeta^2} + \frac{3}{4\zeta^4} + \ldots\right)$$

asymptotically for large argument; and I_n is the Bessel function of imaginary argument of nth order,

$$I_n(z) \sim (z/2)^n \left[1 + 0(z^2)\right] \quad .$$

Corrections to both $\epsilon_\parallel$ and $\epsilon_\perp$ from the next terms in the expansion of Z and of I_n are smaller by $\sim \left(k_\parallel \nu_T/\omega\right)^2$ or $\left[k_\parallel \nu_T/\left(\omega - \Omega_e\right)\right]^2$ and $\sim \left(k_\perp \nu_T/\Omega_e\right)^4$, respectively.

Corrections from higher harmonics ($|n| > 1$) are exponentially small unless $n\Omega_e \approx \omega$ for some n; these are still negligible since we are examining only modes propagating along the axis of the system ($k_\perp \approx 0$).

Substituting Eqs. (2) and (4) in Eq. (1), we obtain

$$\frac{\partial}{\partial x}\left(\epsilon_\perp \frac{\partial \Psi}{\partial x}\right) - k_\parallel^2 \epsilon_\parallel \Psi = 0 \quad . \tag{7}$$

Solution of this equation for some given density profile n(x) in the limit $T \rightarrow 0$ would yield the dispersion relation and analytic form of $\Psi(x)$ for the undamped modes which can propagate; however, difficulties of a mathematical nature prevent us from carrying out the calculation.

It is nevertheless possible to utilize Eq. (7) in a variational calculation which gives the damping rate γ. To do this, rewrite Eq. (7) in the approximate form

$$L\Psi = (L_o + L_1)\,\Psi = f(\omega)\Psi \quad . \tag{8}$$

Here,
$$f(\omega) = k_{\parallel}^{\,2}\left(1 - \omega_p^{\,2}/\omega^2\right) ,$$

and we write $\omega = \omega_o + i\gamma$, $\gamma/\omega_o << 1$. L_o and L_1 are the real and imaginary parts of L:

$$L_o = \frac{d}{dx}\,\epsilon_{\perp}^{\,o}\,\frac{d}{dx} \quad , \qquad \epsilon_{\perp}^{\,o} = 1 - \frac{\omega_p^{\,2}}{\omega_o^{\,2} - \Omega_e^{\,2}} \quad ,$$

and

$$L_1 = i\,\frac{d}{dx}\,\epsilon_{\perp}^{\,1}\,\frac{d}{dx} - i\,k_{\parallel}^{\,2}\,\epsilon_{\parallel}^{\,1} \quad ;$$

$$\epsilon_{\perp}^{\,1} = 2\,\frac{\gamma\omega_o\omega_p^{\,2}}{\left(\omega_o^{\,2} - \Omega_e^{\,2}\right)^2} + \frac{1}{2}\,\frac{\sqrt{\pi}\,\omega_p^{\,2}}{\omega_o k_{\parallel}\nu_T}\left\{\exp\left[-\left(\frac{\omega_o - \Omega_e}{k_{\parallel}\nu_T}\right)^2\right] + \exp\left[-\left(\frac{\omega_o + \Omega_e}{k_{\parallel}\nu_T}\right)^2\right]\right\}$$

$$\epsilon_{\parallel}^{\,1} = 2\sqrt{\pi}\,\frac{\omega_p^{\,2}\,\omega_o}{(k_{\parallel}\nu_T)^3}\,\exp\left[-\left(\frac{\omega_o}{k_{\parallel}\nu_T}\right)^2\right] .$$

Now consider the equation

$$L_o\Psi_o = f(\omega_o)\,\Psi_o \quad , \tag{9}$$

which describes the potential function of undamped waves propagating at T = 0. Multiply Eq. (8) by $\Psi_o^{\,*}(X)$, Eq. (9) by $\Psi^{\,*}(X)$, integrate over x from $-\infty$ to $+\infty$:

$$\int_{-\infty}^{\infty} dx\,\Psi_o^{\,*}L_o\Psi + \int_{-\infty}^{\infty} dx\,\Psi_o^{\,*}L_1\Psi = \int_{-\infty}^{\infty} dx\,f(\omega)\,\Psi_o^{\,*}\Psi \quad , \tag{10}$$

$$\int_{-\infty}^{\infty} dx\,\Psi^{*}L_o\Psi_o = \int_{-\infty}^{\infty} dx\,f(\omega_o)\,\Psi^{*}\Psi_o \quad . \tag{11}$$

Since L_o is self-adjoint,

$$\int_{-\infty}^{\infty} dx\, \Psi_o^{*} L_o \Psi = \int_{-\infty}^{\infty} dx\, \Psi^{*} L_o \Psi_o \quad . \tag{12}$$

We can use the smallness of γ to write

$$f(\omega) - f(\omega_o) \approx (\omega - \omega_o) \frac{\partial f}{\partial \omega} (\omega_o) = i\, \gamma \frac{\partial f}{\partial \omega} (\omega_o) \quad . \tag{13}$$

Subtracting Eq. (11) from Eq. (10), using Eqs. (12) and (13), and approximating $\Psi_o \approx \Psi$, we find

$$i\gamma \int_{-\infty}^{\infty} dx \frac{\partial f}{\partial \omega} (\omega_o) \, |\Psi(x)|^2 = \int_{-\infty}^{\infty} dx\, \Psi^{*}(x)\, L_1 \Psi(x) \quad . \tag{14}$$

Write

$$\frac{\partial f}{\partial \omega}\Big|_{\omega_o} = k_{\|}^2 \left(\frac{2\omega_p^2}{\omega_o^3} \right) ,$$

and integrate by parts in the L_1 term to get

$$\gamma \int_{-\infty}^{\infty} dx \frac{2\omega_p^2 k_{\|}^2}{\omega_o^3} \, |\Psi(x)|^2 = -\int_{-\infty}^{\infty} dx \left(\left\{ \frac{\omega_p^2}{\left(\omega_o^2 - \Omega_e^2\right)^2} 2\gamma\omega_o \right. \right.$$

$$+ \frac{1}{2} \frac{\sqrt{\pi}\,\omega_p^2}{\omega_o k_{\|} \nu_T} \left\{ \exp\left[-\left(\frac{\omega_o + \Omega_e}{k_{\|} \nu_T} \right)^2 \right] \right.$$

$$\left. \left. + \exp\left[-\left(\frac{\omega_o + \Omega_e}{k_{\|} \nu_T} \right)^2 \right] \right\} \right\} \left|\frac{d\Psi}{dx}\right|^2$$

$$\left. + 2\sqrt{\pi}\, k_{\|}^2 \frac{\omega_p^2 \omega_o}{(k_{\|} \nu_T)^3} \exp\left[-\left(\frac{\omega_o}{k_{\|} \nu_T} \right)^2 \right] |\Psi(x)|^2 \right) . \tag{15}$$

This may be solved for γ as

$$\frac{\gamma}{\omega_o} = - \frac{\frac{1}{4}\frac{\sqrt{\pi}\,\omega_o}{k_{\parallel}\nu_T}\left\{\exp\left[-\left(\frac{\omega_o - \Omega_e}{k_{\parallel}\nu_T}\right)^2\right] + \exp\left[-\left(\frac{\omega_o + \Omega_e}{k_{\parallel}\nu_T}\right)^2\right]\right\} F + \sqrt{\pi}\,\frac{\omega_o^{\,3}}{k_{\parallel}^{\,3}\nu_T^{\,3}}\exp\left[-\left(\frac{\omega_o}{k_{\parallel}\nu_T}\right)^2\right] G}{\frac{\omega_o^{\,4}}{\left(\omega_o^{\,2} - \Omega_e^{\,2}\right)^2} F + G} , \qquad (16)$$

where

$$F = \int_{-\infty}^{\infty} dx\ \omega_p^{\,2}\ \left|\frac{d\Psi}{dx}\right|^2 , \qquad (17)$$

$$G = k_{\parallel}^{\,2}\int_{-\infty}^{\infty} dx\ \omega_p^{\,2}\ |\Psi(x)|^2 . \qquad (18)$$

The second exponential which occurs in $\epsilon_{\perp}$ is exponentially smaller than the first, unless

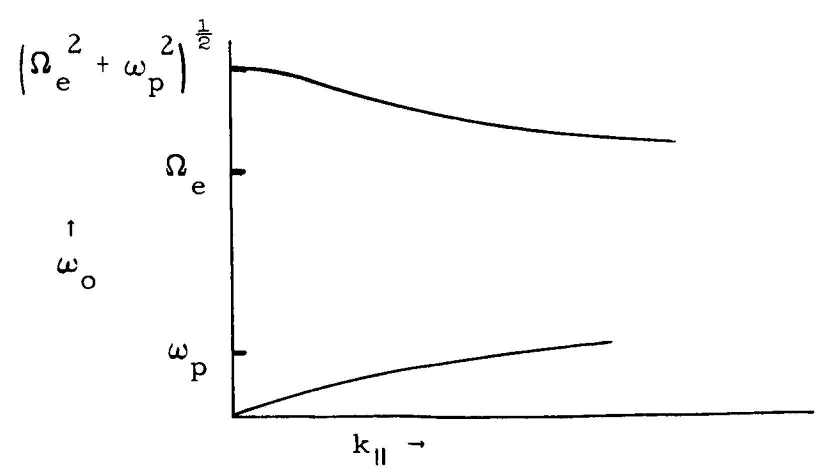
$$\frac{2\omega_o\Omega_e}{(k_{\parallel}\nu_T)^2} \lesssim 1 ,$$

or, by Eq. (3),

$$k_{\parallel} R_e \gtrsim \frac{\omega_o}{k_{\parallel}\nu_T} >> 1 ,$$

where R_e is the electron Larmor radius. We assume this is not the case and drop the term.

For an approximately homogeneous plasma in which $\Omega_e >> \omega_p$, the dispersion curve looks like



On the lower brancn, $\omega_o << \Omega_e$, and the equation for γ reduces to

$$\frac{\gamma}{\omega_o} = - \frac{\dfrac{\sqrt{\pi}\,\omega_o^3}{(k_{\parallel} v_T)^3} \exp\left[-\left(\dfrac{\omega_o}{k_{\parallel} v_T}\right)^2\right] G}{\left(\dfrac{\omega_o}{\Omega_e}\right)^4 F + G} . \tag{19}$$

On the upper branch, $\omega_o \sim \Omega_e$

$$\frac{\gamma}{\omega_o} = - \frac{\dfrac{1}{4}\dfrac{\sqrt{\pi}\,\omega_o}{k_{\parallel} v_T} \exp\left[-\left(\dfrac{\omega_o - \Omega_e}{k_{\parallel} v_T}\right)^2\right] F}{\dfrac{\omega_o^4}{\left(\omega_o^2 - \Omega_e^2\right)^2} F + G} . \tag{20}$$

If $\omega_p >> \Omega_e$ through the bulk of the plasma, there are still two branches; on the lower one, $\omega_o \lesssim \Omega_e$, and

$$\frac{\gamma}{\omega_o} = - \frac{\dfrac{1}{4}\dfrac{\sqrt{\pi}\,\omega_o}{k_{\parallel} v_T} \exp\left[-\left(\dfrac{\omega_o - \Omega_e}{k_{\parallel} v_T}\right)^2\right] F}{\dfrac{\omega_o^4}{\left(\omega_o^2 - \Omega_e^2\right)^2} F + G} . \tag{21}$$

On the upper branch, $\omega_o >> \Omega_e$, and Eq. (16) does not simplify.

In case (b), we assume the potential is azimuthally symmetric:

$$\varphi = \Psi(r) e^{ik_{\parallel} z - i\omega t} \tag{22}$$

and Eq. (7) becomes

$$\frac{1}{r}\frac{d}{dr}\left(r\epsilon_{\perp}\frac{d\Psi}{dr}\right) - k_{\parallel}^2 \epsilon_{\parallel} \Psi = 0 . \tag{23}$$

Here ϵ and $\epsilon_{\perp}$ are again given by Eqs. (5) and (6), since in three dimensions $\underset{\approx}{\epsilon}$ is diagonal,and $\epsilon_{yy} \approx \epsilon_{xx} = \epsilon_{\perp}$ in the approximation (3).

The variational calculation goes through unchanged, except that now

$$L_o = \frac{d}{dr}\left(r\epsilon_\perp{}^o \frac{d}{dr}\right) ,$$

$$L_1 = i\left[\frac{d}{dr}\left(r\epsilon_\perp{}^1 \frac{d}{dr}\right) - k_{\parallel}{}^2\epsilon_{\parallel}{}^1\right] , \quad f(\omega) = k_{\parallel}{}^2 r\left(1 - \frac{\omega_p{}^2}{\omega^2}\right) .$$

The result is identical to Eq. (16), where, however, we now must replace Eqs. (17) and (18) by

$$F' = \int_0^\infty dr\, r\omega_p{}^2 \left|\frac{d\Psi}{dr}\right|^2 , \tag{24}$$

$$G' = k_{\parallel}^2 \int_0^\infty dr\, r\omega_p{}^2 \,|\Psi|^2 . \tag{25}$$

The formula for γ simplifies just as before, when we specialize considerations.

Case (b) may be generalized without difficulty to include azimuthal dependence in Ψ.

One final word of caution should be added. In replacing ω by ω_o in the exponentials, it has been assumed that $\gamma/k_{\parallel}\nu_T \ll k_{\parallel}\nu_T/\omega_o$. If this inequality does not hold, then the replacement is invalid, and Eq. (16) becomes transcendental and γ is in general complex: $\mathrm{Im}(\omega - \omega_o)$ is the damping rate and $\mathrm{Re}(\omega - \omega_o)$ is a frequency shift.

Appendix II

REFERENCES

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2. Fried, B. D., and S. D. Conte, The Plasma Dispersion Function, (Academic Press, Inc., New York, 1961).

APPENDIX III

LANDAU GROWTH RATE FROM ELECTRON BEAM WITH RADIALLY DEPENDENT ENERGY

We imagine that a low temperature beam of electrons moving parallel to $\vec{B}$ is superposed on the Maxwellian which is considered in Appendix II. If we ignore thermal motion within the beam altogether and assume that it is cylindrically symmetric, the part of the electron distribution function arising from the beam has the form

$$f_b(v_\parallel, r) = D(r)\delta\left[v_\parallel - v_o - \alpha(r)\right] \quad , \tag{1}$$

where $\alpha(r) + v_o$ is the velocity with which particles at a distance r from the axis of the system are moving; we assume $\alpha(0) = 0$. In general, $\alpha(r)$ will increase with r, since we may imagine the beam to have arisen as a result of shooting electrons from an electron gun into a potential profile something like that shown in Figure 1.

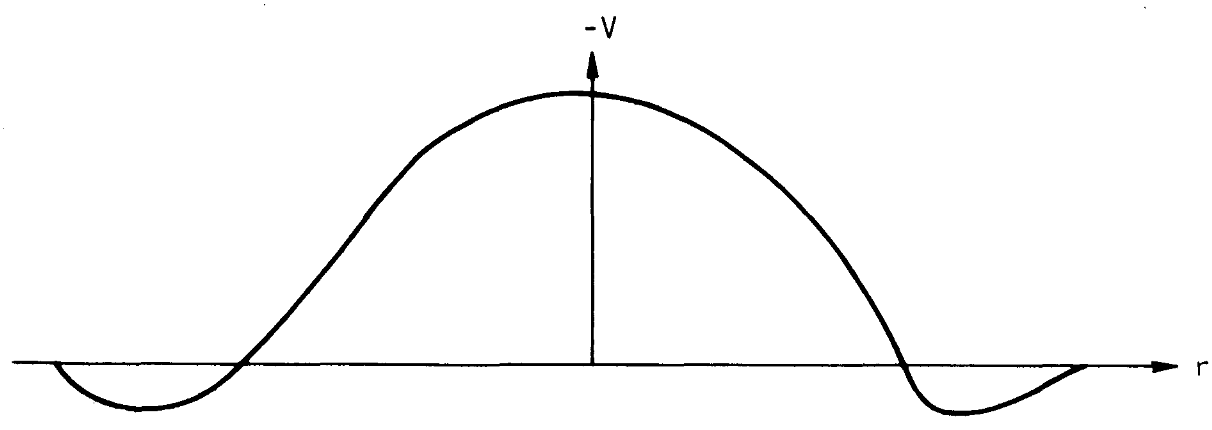

Figure 1

We are primarily interested in simple modes which propagate through the plasma as a whole, that is, in which all the electrons in a cross-sectional slice participate. This being the case, we may expect that for a beam distribution function which varies radially as does Eq. (1), the wave will see an effective distribution averaged over r. In the case of interest, the local beam density is much smaller than the density of the background Maxwellian, and

$$D(r) << \frac{n(r)}{n(0)} = \frac{\omega_p^2(r)}{\omega_p^2(r=0)} \quad . \tag{2}$$

To include the effects of the beam, we retrace the argument leading to Eq. (16) of Appendix II. If we assume $\omega_o << \Omega$, so that only the lower branch contributes to γ, the equation becomes

$$\frac{\gamma}{\omega_o} = \frac{1}{G'}\sqrt{\pi}\,\frac{\omega_o^3}{(k_\parallel v_T)^3}\exp\left[-\left(\frac{\omega_o}{k_\parallel v_T}\right)^2\right]G' \quad , \tag{3}$$

where

$$G' = \int_o^\infty r\,dr\,\omega_p^2 |\psi|^2 \quad .$$

Equation (3) reduces to the usual expression for linear Landau damping, since G' cancels. However, if we write Eq. (3) for a general distribution function, it takes the form

$$\frac{\gamma}{\omega_o} = -\frac{1}{G'}\int_o^\infty r\,dr\,|\psi|^2\,\frac{1}{2}\,\omega_o^2\ \mathrm{Im}\,\epsilon_\parallel \quad , \tag{4}$$

where $\epsilon_\parallel$ is the usual dielectric function

$$\epsilon_\parallel = 1 + \frac{\omega_p^2}{k^2}\int dv_\parallel \frac{\partial f/\partial v_\parallel}{\omega/k - v_\parallel} \quad .$$

When we write $f = f_m + f_{beam}$, the contribution of the beam to the growth (decay) rate is, for $\gamma << \omega_o$,

$$\left(\frac{\gamma}{\omega_o}\right)_{beam} = \frac{1}{G'} \int_0^\infty r\, dr\, |\psi|^2\, \omega_p^{\,2}\, \frac{\pi}{2}\, \omega_o^{\,2} \int_{-\infty}^{\infty} dv\, \frac{\partial g}{\partial v}\, \delta\left(v - \frac{\omega_o}{k}\right), \tag{5}$$

where

$$g = D(r)\, \delta\left[v - v_o - \alpha(r)\right] .$$

In calculating $1/G'$, we can ignore f_b, since the beam density is much smaller than the background density.

Equation (5) may be rewritten in order to investigate particular choices of $\alpha(r)$ and $D(r)$. To do this we assume that $\alpha(r)$ is monotone increasing, so that

$$\delta\left[v - v_o - \alpha(r)\right] = \frac{1}{\alpha'(r_o)}\, \delta(r - r_o) ,$$

where $\alpha(r_o) = v - v_o$ and $\alpha' = d\alpha/dr$. Then integrating by parts we have

$$\left(\frac{\gamma}{\omega_o}\right)_{beam} = \frac{1}{G'} \int_0^\infty dr\, r\, |\psi|^2\, \omega_p^{\,2}\, \frac{\pi}{2} \left(\frac{\omega_o}{k}\right)^2 \int_{-\infty}^{\infty} dv\, \delta\left(v - \frac{\omega_o}{k}\right) D(r)\, \frac{\partial}{\partial v}\left[\frac{1}{\alpha'(r_o)}\, \delta(r - r_o)\right]$$

$$= \frac{1}{G'}\, \frac{\pi}{2} \left(\frac{\omega_o}{k}\right)^2 \int_0^\infty dr\, r\, |\psi|^2\, \omega_p^{\,2}\, D(r) \int_{-\infty}^{\infty} dv\, \frac{\partial}{\partial v}\, \delta\left(v - \frac{\omega_o}{k}\right) \frac{\delta(r - r_o)}{\alpha'(r_o)}$$

$$= -\frac{1}{G'}\, \frac{\pi}{2} \left(\frac{\omega_o}{k}\right)^2 \int_{-\infty}^{\infty} dv\, \frac{\partial}{\partial v}\, \delta\left(v - \frac{\omega_o}{k}\right) \left[\frac{r\, |\psi|^2\, \omega_p^{\,2}\, D(r)\, \theta(r_o)}{\alpha'(r_o)}\right]_{r = r_o}$$

$$= \frac{1}{G'} \frac{\pi}{2} \left(\frac{\omega_o}{k}\right)^2 \frac{\partial}{\partial v} \left\{ \left[\frac{r|\psi|^2 \omega_p^2 D(r)\theta(r_o)}{\alpha'(r_o)} \right]_{r=r_o} \right\}_{v = \frac{\omega_o}{k}} .$$

We can write

$$\frac{\partial}{\partial v} = \left.\frac{\partial r}{\partial v}\right|_{r_o} \frac{\partial}{\partial r} = \frac{1}{\alpha'(r_o)} \frac{\partial}{\partial r} ,$$

so

$$\left(\frac{\gamma}{\omega_o}\right)_{\text{beam}} = \frac{1}{G'} \frac{\pi}{2} \left(\frac{\omega_o}{k}\right)^2 \frac{1}{\alpha'(r_o)} \frac{\partial}{\partial r} \left[\frac{r|\psi|^2 \omega_p^2 D(r)\theta(r)}{\alpha'(r)} \right]_{r = \alpha^{-1}\left(\frac{\omega_o}{k} - v_o\right)}$$

$$= \frac{1}{G'} \frac{\pi}{2} \left(\frac{\omega_o}{k}\right)^2 \frac{1}{\alpha'(r_o)} \left\{ \frac{r_o|\psi|^2 \omega_p^2 D(r_o)\delta(r_o)}{\alpha'(r_o)} + \theta(r_o) \cdot \frac{\partial}{\partial r} \left[\frac{r|\psi|^2 \omega_p^2 D(r)}{\alpha'(r)} \right] \right\}_{r = \alpha^{-1}\left(\frac{\omega_o}{k} - v_o\right) = r_o} . \qquad (6)$$

This shows that the contribution to γ for a particular phase velocity ω_o/k comes from particles at a distance r_o sufficient to make

$$v_o + \alpha(r_o) = \frac{\omega_o}{k} .$$

For $\omega_o/k < v_o$, there are no such particles, since $\alpha(r) > 0$; so if $rD(r)/\alpha'(r) > 0$ as $r \to 0$, there is a jump in the effective velocity distribution, yielding a δ function dependence in γ. This δ function results from our assumption that the beam had zero temperature; as will be seen below, it is smoothed out for finite beam temperatures.

Following Figure 1, let us examine some plausible choices for $\alpha(r)$. We assume $D(r) = D$, a constant, and assume that ω_p^2 and $|\psi|^2$ are roughly constant out to some value $r = R$.

The first choice is

$$\alpha(r) = Ar^2, \qquad A = \text{constant}. \tag{7}$$

Then $\alpha'(r) = 2Ar$, and $r_o = (v - v_o/A)^{\frac{1}{2}}$. We see that there is a jump in the effective distribution, since

$$\lim_{r \to 0+} \left[\frac{r}{\alpha'(r)}\right] = \frac{1}{2A} = \text{const.}$$

Formula (6) yields

$$\left(\frac{\gamma}{\omega_o}\right)_{\text{beam}} = \frac{1}{G'} \frac{\pi}{2} \left(\frac{\omega_o}{k}\right)^2 \frac{1}{2\left[A(v-v_o)\right]^{\frac{1}{2}}} \frac{D|\psi|^2 \omega_p^2}{2A} \left[\delta(r_o) - \delta(r_o - R)\right] . \tag{8}$$

This growth rate is shown in Figure 2.

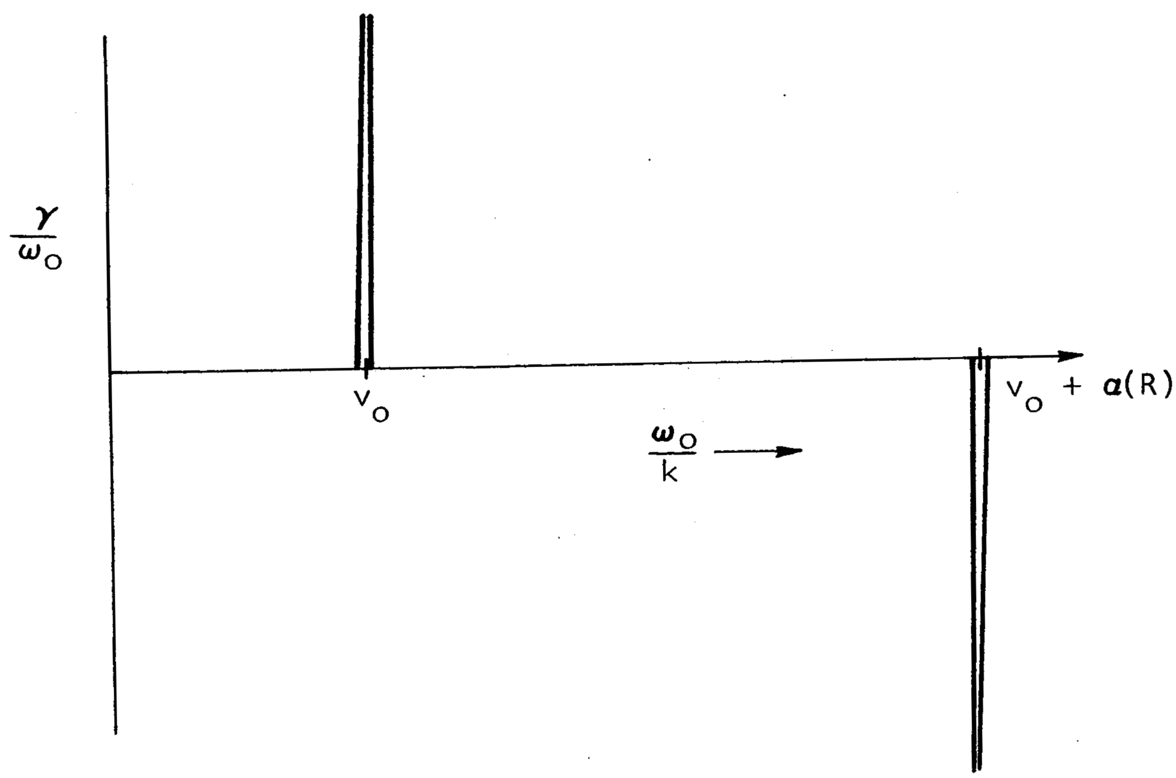

Figure 2

The peak at $r_o = R$ is of course rounded if we retain a realistic radial dependence for $\omega_p{}^2 |\psi|^2$.

For the second choice of $\alpha(r)$, we argue as follows: an electron will have kinetic energy

$$\frac{1}{2} mv^2 = E_o - V \ , \tag{9}$$

where E_o is the "muzzle" energy of the electron gun and V is shown in Figure 1. If V has roughly parabolic dependence on r, then

$$v = \left(\frac{2E_o}{m} - \frac{2V}{m}\right)^{\frac{1}{2}}$$

$$\approx \left(v_o{}^2 + \nu^2 r^2\right)^{\frac{1}{2}} \ . \tag{10}$$

Thus

$$\alpha(r) = (v_o{}^2 + \nu^2 r^2)^{\frac{1}{2}} - v_o \ ;$$

$$\alpha'(r) = \frac{\nu^2 r}{(v_o{}^2 + \nu^2 r^2)^{\frac{1}{2}}} \ ;$$

$$\frac{d}{dr}\left[\frac{r}{\alpha'(r)}\right] = \frac{r}{(v_o{}^2 + \nu^2 r^2)^{\frac{1}{2}}} \ ;$$

$$\frac{1}{\alpha'(r)} \frac{d}{dr}\left[\frac{r}{\alpha'(r)}\right] = \frac{1}{\nu^2} \ .$$

Now Formula (6) tells us that for $0 < r_o < R$,

$$\left(\frac{\gamma}{\omega_o}\right)_{\text{beam}} = \frac{1}{G'} \frac{\pi}{2} \left(\frac{\omega_o}{k}\right)^2 D \, \omega_p{}^2 \, |\psi|^2 \frac{1}{\nu^2} \ .$$

This growth rate is represented in Figure 3.

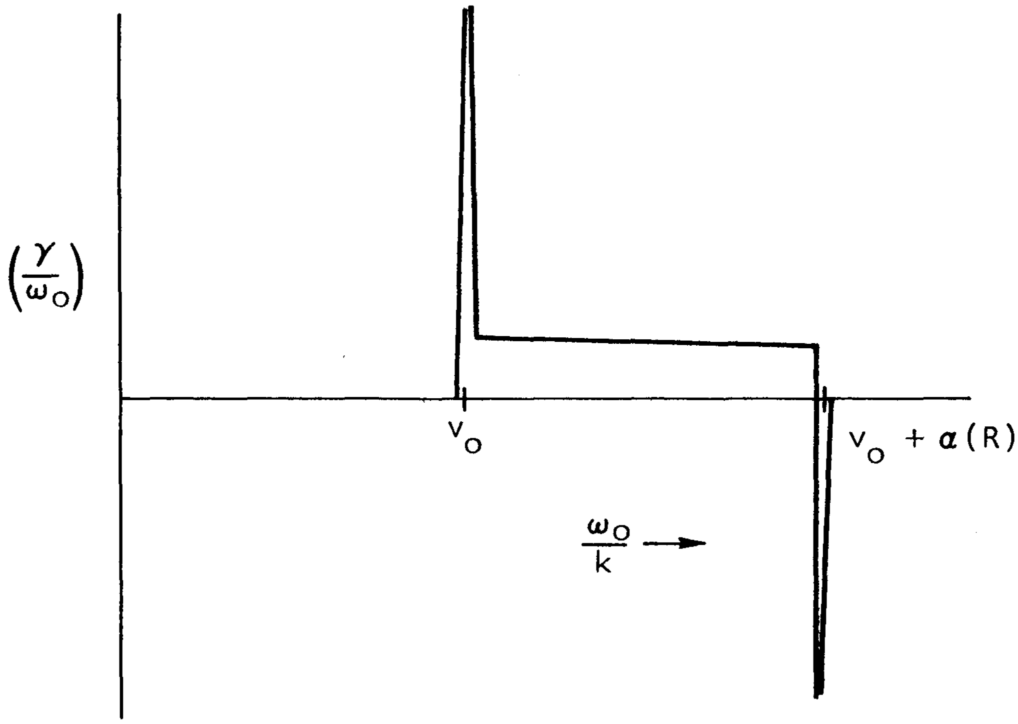

Figure 3

It is desirable to make a beam which leads to slowly varying, bounded values of γ (to which the "gentle bump" quasi-linear theory applies). Let us consider the effect of including finite, though small, beam temperature.

In $f_b + D(r)\delta\left[v - v_o - \alpha(r)\right]$, we replace δ by δ_ϵ, defined by

$$\delta_\epsilon(x) = \frac{1}{\epsilon\sqrt{\pi}} \exp\left[-\left(\frac{x}{\epsilon}\right)^2\right]. \tag{11}$$

We can examine the lowest order correction in ϵ:

$$\int f(x)\delta_\epsilon(x) = \int \frac{1}{\epsilon\sqrt{\pi}} \exp\left[-\left(\frac{x}{\epsilon}\right)^2\right]\left[f(0) + xf'(0) + \frac{1}{2}x^2 f''(0) + \ldots\right]$$

$$\approx f(0) + \frac{\epsilon^2}{4} f''(0) \quad ,$$

so

$$\delta_{\epsilon}(x) \approx \delta(x) + \frac{\epsilon^2}{4}\,\delta''(x) \quad . \tag{12}$$

Again using

$$\delta\left[v - v_o - \alpha(r)\right] = \frac{1}{\alpha'(r_o)}\,\delta(r - r_o) \quad ,$$

the correction term becomes

$$\frac{\epsilon^2}{4}\left[\left(\frac{3\alpha''^2}{\alpha'^5} - \frac{\alpha'''}{\alpha'^4}\right)\delta(r - r_o) + \frac{3\alpha''}{\alpha'^4}\,\delta'(r - r_o) + \frac{1}{\alpha'^3}\,\delta''(r - r_o)\right] \quad .$$

For the first case above, $\alpha = Ar^2$, the correction is

$$-\frac{9}{2}\,\frac{1}{\left(\frac{\omega_o}{k} - v_o\right)^3}\,\frac{\epsilon^2}{4}\,\frac{D}{G'}\,\frac{\pi}{2}\left(\frac{\omega_o}{k}\right)^2 \omega_p^{\,2}\,|\psi|^2 \quad ;$$

for the second example, $\alpha = (v_o^{\,2} + \nu^2 r^2)^{\frac{1}{2}} - v_o$, the correction is

$$\frac{\epsilon^2}{4}\,\frac{D}{G'}\,\frac{\pi}{2}\left(\frac{\omega_o}{k}\right)^2 \omega_p^{\,2}\,|\psi|^2\,\frac{9}{\nu^2}\left[-\,8\left(\frac{\omega_o}{k}\right)^4\left(\frac{\omega_o}{k} - v_o\right)^{-6}\right.$$

$$\left. +\;12\left(\frac{\omega_o}{k}\right)^2\left(\frac{\omega_o}{k} - v_o\right)^{-4} - 6\left(\frac{\omega_o}{k} - v_o\right)^{-2} - \left(\frac{\omega_o}{k}\right)^{-2}\right] \quad .$$

Both corrections are negative and blow up at $\omega_o/k = v_o$, thus tending to cancel the δ-function at $r_o = 0$.

It is not necessary to treat finite beam temperature to remove the peak in γ. Thus far we have taken $D(r)$ to be a constant. If a template is placed before the stream of electrons from the electron gun, it can screen some out, so that $D(r)$ may vanish at $r = 0$. For example, let the template have the form shown in Figure 4.

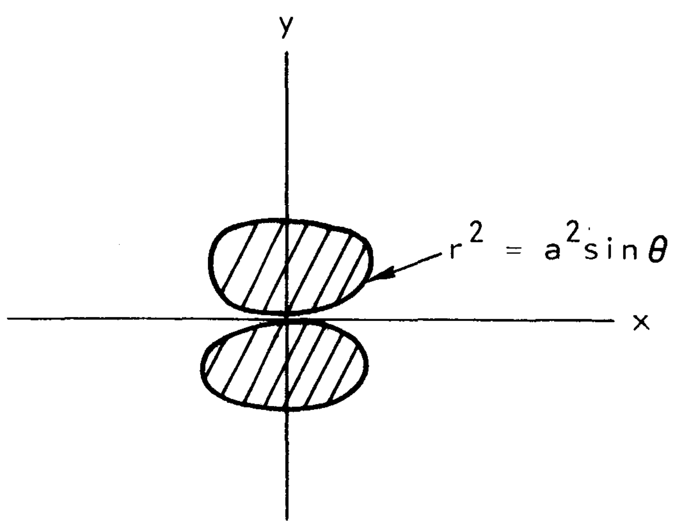

Figure 4

Then $D(r) \sim 4 \int_0^{\sin^{-1} \frac{r^2}{a^2}} d\theta \approx 4 \frac{r^2}{a^2}$ for r small.

Now, if $\omega_p^2 |\Psi|^2$ gradually decreases with the increase of r (instead of sharply dropping at $r = R$), we have for γ a curve of the form

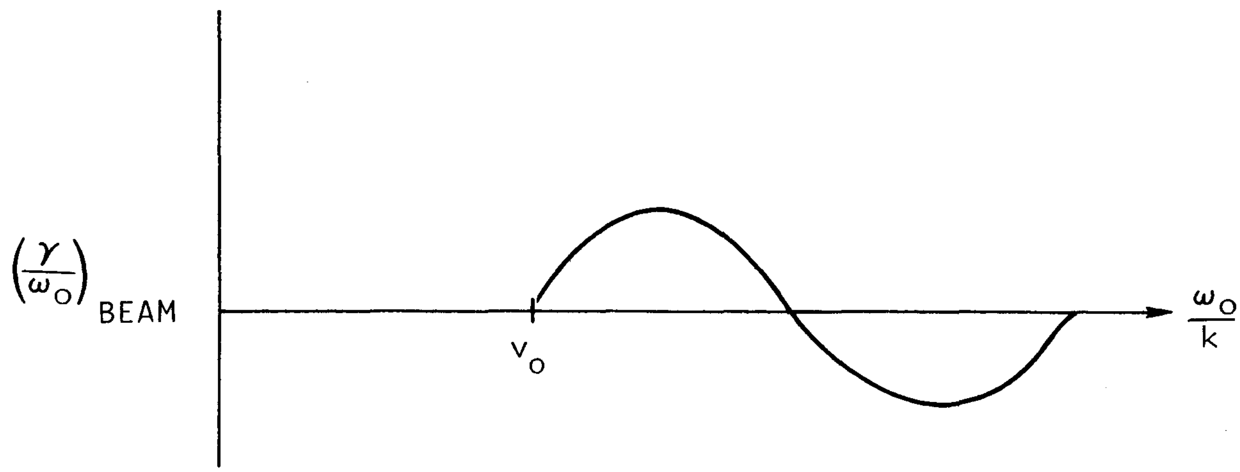

Figure 5

Note that the area under the curve sums to zero. This follows from Formula (6), which expresses γ as a total derivative with respect to v of a quantity which vanishes as $v \to v_o$ and $v \to \infty$.

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13. ABSTRACT

The dispersion relations for the electrostatic waves in collisionless plasma have been measured near the electron cyclotron frequency. Several waves are observed including the C_{01} backward wave between the cyclotron frequency and the upper hybrid frequency and the plasma perturbed T_e wave guide mode. The cyclotron waves are heavily damped. The experimental techniques necessary for the measurement are described. The Landau damping of the electrostatic modes of a plasma with finite radial dimensions is investigated theoretically. The growth of these electrostatic modes due to a beam whose energy depends on radius is also calculated.

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INSTRUCTIONS

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